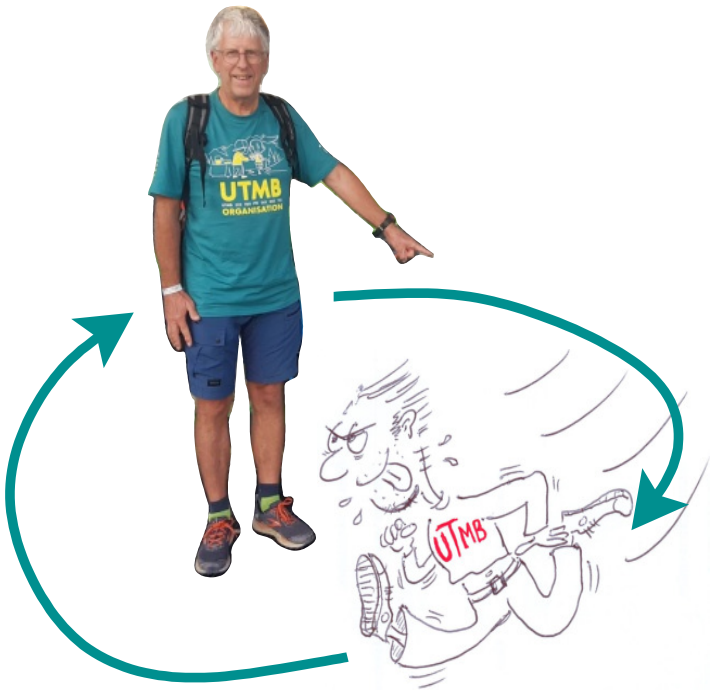
Correlations of occupation numbers in the canonical ensemble

Christophe Texier

LPTMS
Laboratoire de Physique Théorique et Modèles Statistiques

université PARIS-SACLAY



with **Olivier Giraud** (LPTMS) & **Aurélien Grabsch** (LPTMC, SU)

Table of contents

1. Introduction and motivations

2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

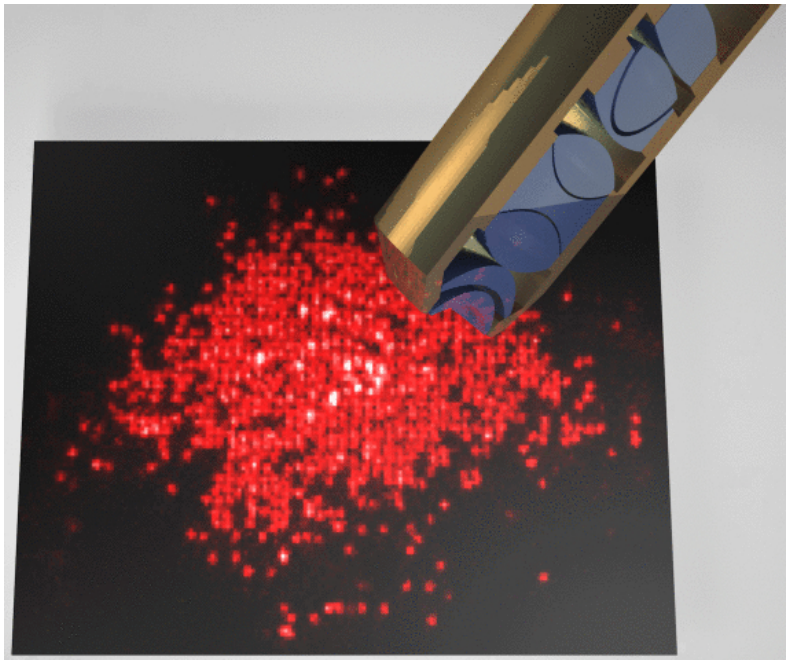
Plan

1. Introduction and motivations

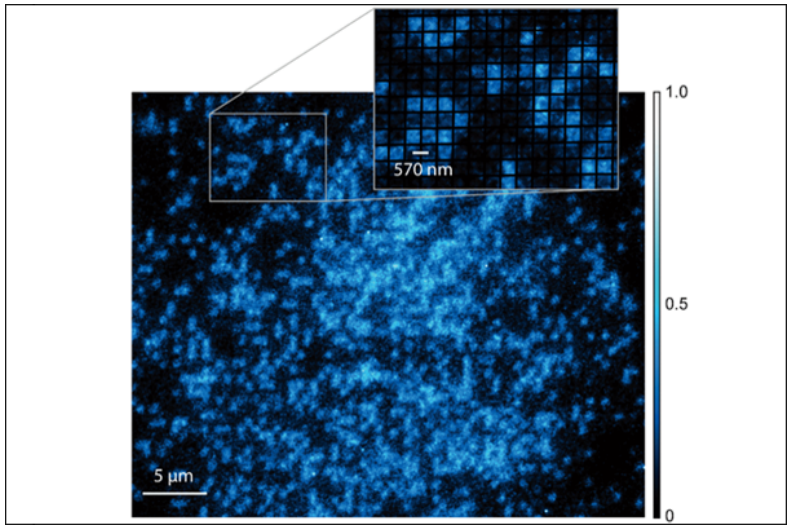
2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

Fermi microscope



I. Bloch's group (München)

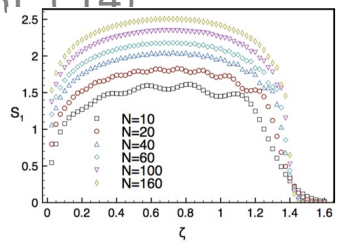


M. Greiner's group (Harvard)

Fermi gas (study quantum correlations)

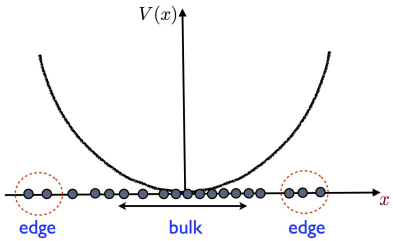
- $T = 0$ for 1D harmonically trapped fermions ($\leftrightarrow$ GβE of RMT)
  - Edge correlations, Eissler, PRL (2013)
  - index : $\mathcal{N}_+ = \sum_i \theta_{\rm H}(x_i)$, Majumdar, Marino, Schehr, Vivo, PRL ('14)
  - entanglement entropy :

  Calabrese, Le Doussal, Majumdar, PRA **91** (2015) ; ....



- Correlation in bulk *vs* at edge ; $T = 0$ *vs* $T > 0$; $d = 1$ and $d > 1$
  Dean, Le Doussal, Majumdar, Schehr, PRA **94** (2016)

- Hard wall confinement :
  B. Lacroix-à-chez-Toine, Majumdar, Schehr, EPL (2017)

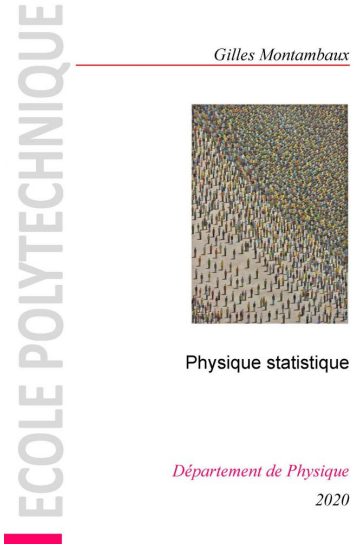


- Observables in the grand-canonical AND canonical ensembles
  Grela, Majumdar, Schehr, PRL **119** (2018) ;
  Grabsch, Majumdar, Schehr, CT, SciPost **4** (2018)

  $\overline{\mathcal{O}}^{\rm g} \simeq \overline{\mathcal{O}}^{\rm c}$: equivalence between ensembles

  $\mathrm{Var_g}(\mathcal{O}) \simeq \mathrm{Var_c}(\mathcal{O}) + \mathrm{Var_g}(N) \left(\frac{\partial \overline{\mathcal{O}}^{\rm c}}{\partial N}\right)^2 \xrightarrow{\text{canonical}}$ need $\overline{n_\lambda}^{\rm c}$ & $\overline{n_\lambda n_\mu}^{\rm c}$

Grand canonical ensemble



(undergraduate)

- Individual states : $\{|\lambda\rangle, \varepsilon_\lambda\}$
- Fugacity : $\varphi = e^{\beta\mu}$
- Mean occupations (BE/FD) :

$$\overline{n_\lambda}^{\mathrm{g}} = \frac{1}{e^{\beta\varepsilon_\lambda}/\varphi \mp 1}$$

- Occupations of individual states are

INDEPENDENT

$$\overline{n_{\lambda_1}\cdots n_{\lambda_p}}^{\mathrm{g}} = \overline{n_{\lambda_1}}^{\mathrm{g}} \times \overline{n_{\lambda_2}}^{\mathrm{g}} \times \cdots \times \overline{n_{\lambda_p}}^{\mathrm{g}}$$

Question :

$$\overline{n_\lambda}\ ,\ \overline{n_{\lambda_1} n_{\lambda_2}}\ ,\ \ldots\ ,\ \overline{n_{\lambda_1} n_{\lambda_2} \cdots n_{\lambda_p}}$$

in the **canonical** ensemble ?

Plan

1. Introduction and motivations

2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

Plan

1. Introduction and motivations

2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

Quantum mechanics : Quantum states are $|\{n_\lambda\}\rangle$

n_λ = # particles in *individual* state $|\lambda\rangle$ of energy ε_λ

$$\boxed{x_\lambda \overset{\text{def}}{=} e^{-\beta\varepsilon_\lambda}}$$

Canonical ensemble (N fixed) :

$$\mathscr{P}_N^{\text{c}}(\{n_\lambda\}) \propto e^{-\beta E(\{n_\lambda\})} = e^{-\beta\sum_\lambda n_\lambda\varepsilon_\lambda} = \prod_\lambda x_\lambda^{n_\lambda}$$

$$\mathscr{P}_N^{\text{c}}(\{n_\lambda\}) = \frac{\prod_\lambda x_\lambda^{n_\lambda}}{Z_N}\,\delta_{N,\sum_\lambda n_\lambda}\,, \qquad Z_N = \sum_{\{n_\lambda\}}\delta_{N,\sum_\lambda n_\lambda}\prod_\lambda x_\lambda^{n_\lambda}$$

Grand canonical ensemble (μ fixed) ; $\boxed{\varphi = e^{\beta\mu}}$

$$\mathscr{P}_\varphi^{\text{g}}(\{n_\lambda\}) = \frac{\prod_\lambda (x_\lambda\varphi)^{n_\lambda}}{\Xi_\varphi}\,, \qquad \Xi_\varphi = \sum_{\{n_\lambda\}}\prod_\lambda (x_\lambda\varphi)^{n_\lambda} = \overbrace{\prod_\lambda \xi_\lambda}^{\text{Factorizes}}$$

Average of an observable $\mathcal{O}$

$$\Xi_\varphi\,\overline{\mathcal{O}(\{n_\lambda\})}^{\text{g}} = \sum_{N=0}^{\infty} Z_N\,\varphi^N\,\overline{\mathcal{O}(\{n_\lambda\})}^{\text{c},\,(N)}$$

Plan

1. Introduction and motivations

2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

Starting point : $\displaystyle\sum_{N=0}^{\infty} Z_N\,\varphi^N\,\overline{\mathcal{O}(\{n_\lambda\})}^{\mathrm{c},\,(N)} = \Xi_\varphi\,\overline{\mathcal{O}(\{n_\lambda\})}^{\mathrm{g}}$

Application : $\mathcal{O}(\cdot) \to n_\lambda$

$$\sum_{N=1}^{\infty} Z_N\,\varphi^N\,\overline{n_\lambda} = \Xi_\varphi\,\overbrace{\frac{1}{1/(x_\lambda\varphi)\mp 1}}^{\overline{n_\lambda}^{\mathrm{g}}} = \pm\sum_{m=0}^{\infty} Z_m\,\varphi^m \sum_{k=1}^{\infty}(\pm x_\lambda\,\varphi)^k$$

term φ^N : $\quad \overline{n_\lambda} = \pm\displaystyle\sum_{k=1}^{N}(\pm x_\lambda)^k\,\frac{Z_{N-k}}{Z_N}$ $\quad (\pm = \text{BE/FD})$

Weiss & Wilkens (1997) ; Chase, Mekjian & Zanick (1999) ; Bormann et al (1999) ;...

Correlations

$$\overline{n_1\cdots n_p} = \frac{(\pm 1)^p}{Z_N}\sum_{\substack{k_1,\,\cdots,\,k_p=1\\ k_1+\cdots+k_p\leqslant N}}^{N}(\pm x_1)^{k_1}\cdots(\pm x_p)^{k_p}\,Z_{N-k_1-\cdots-k_p}$$

- $\overline{n_1^{r_1}\cdots n_p^{r_p}}$ for bosons : add p factors $a_{k_i}(r_i) = k_i^{r_i} - (k_i-1)^{r_i}$

More simple representations ?

Plan

1. Introduction and motivations

2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

Indistinguishability

Canonical ensemble (N fixed)

$$Z_N(x_1, x_2, \cdots, x_\lambda, \cdots) = \sum_{\{n_\lambda\}} \delta_{N, \sum_\lambda n_\lambda} \prod_\lambda x_\lambda^{n_\lambda} \qquad \text{where } x_\lambda \stackrel{\text{def}}{=} e^{-\beta \varepsilon_\lambda}$$

Grand canonical ensemble (μ fixed) ; $\boxed{\varphi = e^{\beta\mu}}$

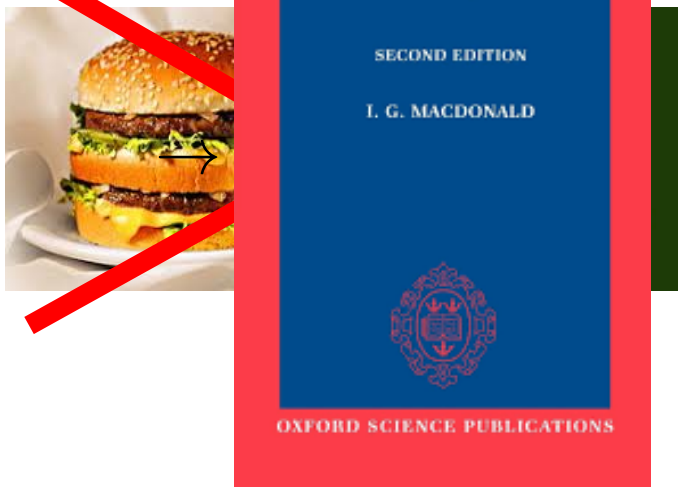
$$\Xi_\varphi(x_1, x_2, \cdots, x_\lambda, \cdots) = \sum_{\{n_\lambda\}} \prod_\lambda (x_\lambda \varphi)^{n_\lambda}$$

Average of an observable $\mathcal{O}$

$$\Xi_\varphi \, \overline{\mathcal{O}(\{n_\lambda\})}^{\mathrm{g}} = \sum_{N=0}^{\infty} Z_N \, \varphi^N \, \overline{\mathcal{O}(\{n_\lambda\})}^{\mathrm{c},\,(N)}$$

These are symmetric functions of $(x_1, x_2, \cdots, x_\lambda, \cdots)$

Theory of symmetric functions : ... the « MacDonald » ...



WIKIPEDIA
The Free Encyclopedia

Article Talk

Symmetric polynomial

From Wikipedia, the free encyclopedia

WIKIPEDIA
The Free Encyclopedia

Article Talk

Newton's identities

From Wikipedia, the free encyclopedia

Consider M variables $x_1,\ x_2,\ \cdots,\ x_M$

$$\phi(x_{\pi(1)},\ x_{\pi(2)},\ \cdots,\ x_{\pi(M)}) = \phi(x_1,\ x_2,\ \cdots,\ x_M) \qquad \forall \text{ permutation } \pi$$

1) Elementary symmetric functions $e_N(x_1, \cdots, x_M) \longrightarrow e_N \equiv Z_N^{\mathrm{F}}(\beta)$

$$e_N(x_1,\ x_2,\ \cdots,\ x_M) \overset{\text{def}}{=} \sum_{1 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_N \leqslant M} x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_N}$$

$$\begin{cases} e_0 = 1 \\ e_1 = x_1 + x_2 + \cdots + x_M \\ e_2 = x_1 x_2 + x_1 x_3 + \cdots + x_2 x_3 + \cdots \\ e_3 = x_1 x_2 x_3 + x_1 x_2 x_4 + \cdots \\ \quad \vdots \quad \vdots \\ e_M = x_1 x_2 \cdots x_M \end{cases}$$

$$\text{Generating fct} : \Xi^{\mathrm{F}}(\varphi) = \sum_{N=0}^{M} \varphi^N e_N \equiv \sum_{N=0}^{M} \varphi^N Z_N^{\mathrm{F}} = \prod_{\lambda=1}^{M} (1 + \varphi\, x_\lambda)$$

2) Complete symmetric functions $h_N(x_1, \cdots, x_M) \longrightarrow h_N \equiv Z_N^{\mathrm{B}}(\beta)$

$$h_N(x_1,\ x_2,\ \cdots,\ x_M) \overset{\text{def}}{=} \sum_{\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_N} x_{\lambda_1} x_{\lambda_2} \cdots x_{\lambda_N}$$

$$\text{Generating fct} : \Xi^{\mathrm{B}}(\varphi) = \sum_{N=0}^{\infty} \varphi^N h_N \equiv \sum_{N=0}^{\infty} \varphi^N Z_N^{\mathrm{B}} = \prod_{\lambda=1}^{M} (1 - \varphi\, x_\lambda)^{-1}$$

Relation between $Z_N^{\mathrm{F}}(\beta)$'s and $Z_N^{\mathrm{B}}(\beta)$'s

$$\Xi^{\mathrm{B}}(\varphi) = \prod_{\lambda} \frac{1}{1 - \varphi\, x_\lambda} \qquad \text{and} \qquad \Xi^{\mathrm{F}}(\varphi) = \prod_{\lambda} (1 + \varphi\, x_\lambda)$$

$$\Xi^{\mathrm{B}}(\varphi) = \frac{1}{\Xi^{\mathrm{F}}(-\varphi)} \quad \Rightarrow \begin{cases} Z_1^{\mathrm{B}} = Z_1^{\mathrm{F}} \\ Z_2^{\mathrm{B}} = (Z_1^{\mathrm{F}})^2 - Z_2^{\mathrm{F}} \\ Z_3^{\mathrm{B}} = (Z_1^{\mathrm{F}})^3 - 2\, Z_1^{\mathrm{F}} Z_2^{\mathrm{F}} - Z_3^{\mathrm{F}} \\ \quad \vdots \quad \vdots \end{cases}$$

3) Power sum polynomials $z_k(x_1,\, x_2,\, \cdots,\, x_M) \longrightarrow z_k \equiv Z_1^{\mathrm{B}}(k\beta) = Z_1^{\mathrm{F}}(k\beta)$

$$z_k(x_1,\, x_2,\, \cdots,\, x_M) \stackrel{\text{def}}{=} \sum_{\lambda=1}^{M} x_\lambda^k \qquad = e_1(x_1^k, \cdots, x_M^k) = h_1(x_1^k, \cdots, x_M^k)$$

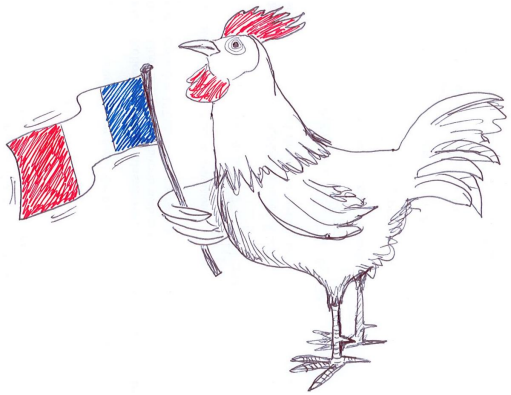
Generating function : $P(\varphi) = \sum_{k=1}^{\infty} \varphi^{k-1} z_k = \sum_{\lambda=1}^{M} x_\lambda/(1-\varphi x_\lambda)$

Newton identity (1666)

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\Xi^{\mathrm{F}}(\varphi) = P(-\varphi)\,\Xi^{\mathrm{F}}(\varphi) \qquad \Rightarrow \qquad \boxed{Z_N^{\mathrm{F}} = \frac{1}{N}\sum_{k=1}^{N}(-1)^{k-1} z_k\, Z_{N-k}^{\mathrm{F}}}$$

Landsberg, *Thermodynamics – with quantum statistical illustrations*, 1961 ; Ford, Am. J. Phys. (1971) ; Borrman & Franke, J. Chem. Phys. (1993) ; ...

... or Albert Girard identity (1629) ?



BnF Gallica

Invention nouvelle
EN
L'ALGEBRE,
PAR
ALBERT GIRARD
MATHEMATICIEN.

ant pour la ſolution des equations, que pour recognoiſtre le nombre des ſolutions qu'elles reçoivent, avec pluſieurs choſes qui ſont neceſſaires à la perfection de ceſte divine ſcience.

A AMSTERDAM.
Chez Guillaume Ianſſon Blaeuw.
M. DC. XXIX.

6920 (2)

A MONSIEVR
Monsr HENRY de BERGAIGNE
Capitaine d'une Compagnie de Cavallerie pour Meſſeig^rs les Eſtats Generaux des Provinces Vnies des Pays Bas, Receveur des contributions de Brabant au quartier de Breda, &c.

MONSIEVR,

Entre les ſciences leſquelles vous cheriſſez, & qui vous ſont familieres, vous n'y avez pas ſimplement rangé les Mathematiques, mais auſſi y avez fait un progrez au delà du vulgaire, laquelle choſe, & principalement la renommée de vos vertus, m'ont acertené que vous recevriez d'un bon œil ces trois petits traictez, dont le premier n'eſt qu'une briefve introduction en l'Arithmetique, mais les deux autres contiennent quelques nouveautez en l'Algebre & Geometrie,

 incogneues non ſeulement des modernes, mais auſſi des anciens, & n'y a autre choſe qui me poind preſentement, ſinon qu'elles ſont un peu trop toſt ſorties de ma main pour leur donner quelque luſtre, & auſſi que jé n'ay quelqu'autre ſubject tout appreſté à vous demonſtrer en effect que je me repute

Monſieur,

Voſtre tres-humble & tres-affectionné ſerviteur
Albert Girard.

COM-

faire incontinent l'equation, quand la queſtion eſt des factions, comme:

Trouvons trois nombres, dont la ſomme ſoit 12, les trois produits de deux à deux 41, leur ſolide 42.

Pource que toutes les factions ſont denommées autant que trois nombres peuvent recevoir, on poſera à part 1 ③ : Et puis la ſomme des nombres 12, avec une quantité moindre ſubſequente 12 ② : Auſſi 41 ①, finalement 42 ⓪, puis les ayant mis d'ordre alternatif, & ce ſelon les ſignes de la propoſition (qui ſont icy tous +) on aura 1 ③ + 41 ① eſgale à 12 ② + 42 : leſquels remis en ordre poſterieur 1 ③ ſera eſgale à 12 ② — 41 ① + 42 : laquelle eſt apareillée pour reſoudre : dont les trois ſolutions ſeront les trois nombres requis, & ainſi s'entendra des ſemblables queſtions.

Il pourroit ſembler à quelqu'un que les factions ſeroyent encor expliquables autrement que deſſus, comme au lieu de dire, la ſomme : les produits de deux à deux ; les produits de trois à trois, &c. qu'on pourroit dire & plus ſimplement : La ſomme : la ſomme des quarez : la ſomme des Cubes, &c. ce qui n'eſt pas ainſi, car ſoyent pluſieurs ſolutions, la ſomme ſera pour le premier meſlé, la ſomme des produits deux à deux pour le ſecond meſlé, &c. comme il a eſté ſuffiſamment expliqué ; mais il n'en eſt pas ainſi des puiſſances comme on pourroit objecter.

Exemple.

Soit
- A premier meſlé.
- B ſecond.
- C troiſieſme.
- D quatrieſme.
- &c.

alors en toute ſorte d'equation.

		ſera la ſomme des
A		ſolutions
A q — B 2		quarez
A cub — A B 3 + C 3		Cubes
A q q — A q B 4 + A C 4 + B q 2 — D 4		quaré-quarez

Et pour mieux expliquer le tout, ſoit 1 ④ + 35 ② + 24 eſgale à 10 ③ + 50 ① : l'ordre des meſlez eſt 10. 35. 50. 24 pour A, B, C, D, cy-deſſus : tellement que 10 eſt voirement la ſomme des ſolutions qui ſont (1, 2, 3, 4.) Or A q — B 2, c'eſt à dire le quarré de 10 — deux fois 35 c'eſt la ſomme des quarrez, & ainſi du reſte ; ſoit auſſi qu'on

prenne

$$P(-\varphi) = \frac{\mathrm{d}}{\mathrm{d}\varphi} \ln \Xi^{\mathrm{F}}(\varphi)$$

$$\Downarrow$$

$$z_1 = Z_1^{\mathrm{F}}$$

$$z_2 = (Z_1^{\mathrm{F}})^2 - 2(Z_2^{\mathrm{F}})$$

$$z_3 = (Z_1^{\mathrm{F}})^3 - 3\, Z_1^{\mathrm{F}}\, Z_2^{\mathrm{F}} + 3\, Z_3^{\mathrm{F}}$$

$$z_4 = (Z_1^{\mathrm{F}})^4 - 4(Z_1^{\mathrm{F}})^2\, Z_2^{\mathrm{F}} + 4\, Z_1^{\mathrm{F}}\, Z_3^{\mathrm{F}} + 2(Z_2^{\mathrm{F}})^2 - 4\, Z_4^{\mathrm{F}}$$

$$z_k \equiv Z_1^{\mathrm{B}}(k\beta) = Z_1^{\mathrm{F}}(k\beta)$$

Relation between $Z_N^{\mathrm{B}/\mathrm{F}}(\beta)$'s and $Z_1(k\beta) \equiv z_k$'s

... or .... (cf. example 8, §2 of chapter 1 of MacDonald)

$$Z_N^{\mathrm{F}} = \frac{1}{N!} \begin{vmatrix} z_1 & 1 & 0 & \cdots & 0 \\ z_2 & z_1 & 2 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & & & N-1 \\ z_N & z_{N-1} & z_{N-2} & \cdots & z_1 \end{vmatrix}$$

... and also

$$Z_N^{\mathrm{B}} = \frac{1}{N!} \begin{vmatrix} z_1 & -1 & 0 & \cdots & 0 \\ z_2 & z_1 & -2 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & & & -N+1 \\ z_N & z_{N-1} & z_{N-2} & \cdots & z_1 \end{vmatrix}$$

A useful relation for $p = 2$

$$\overline{n_1 n_2} = \mp \frac{e^{\beta\varepsilon_1}\overline{n_1} - e^{\beta\varepsilon_2}\overline{n_2}}{e^{\beta\varepsilon_1} - e^{\beta\varepsilon_2}} = \mp \frac{\overline{n_1}x_2 - \overline{n_2}x_1}{x_2 - x_1} = \mp \frac{\begin{vmatrix} \overline{n_1} & x_1 \\ \overline{n_2} & x_2 \end{vmatrix}}{\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}}$$

Valid for • grand canonical : trivial

• canonical : non trivial

Grabsch, Majumdar, Schehr, CT, SciPost Phys. **4** (2018)

General case

$$\overline{n_1 n_2 \cdots n_p} = (\mp 1)^{p-1} \frac{\begin{vmatrix} \overline{n_1} & x_1 & \cdots & x_1^{p-1} \\ \overline{n_2} & x_2 & \cdots & x_2^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{n_p} & x_p & \cdots & x_p^{p-1} \end{vmatrix}}{\begin{vmatrix} 1 & x_1 & \cdots & x_1^{p-1} \\ 1 & x_2 & \cdots & x_2^{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_p & \cdots & x_p^{p-1} \end{vmatrix}}$$

+ other recurrence relations involving smaller determinants

Grabsch, Giraud, CT, PRA **97** (2018)

$\Rightarrow$ p-correlations are expressed in terms of $\overline{n_k}$'s

Schur function s_λ

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ a partition of an integer

(with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ integers)

$$a_\lambda(x_1, \ldots, x_n) \stackrel{\text{def}}{=} \begin{vmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \cdots & x_1^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \cdots & x_n^{\lambda_n} \end{vmatrix}$$

Vandermonde : $\delta \stackrel{\text{def}}{=} (n-1, n-2, \cdots, 2, 1, 0)$

$a_\delta(x_1, \ldots, x_n)$ is the Vandermonde

$$s_\lambda(x_1, \ldots, x_n) = \frac{a_{\lambda+\delta}(x_1, \ldots, x_n)}{a_\delta(x_1, \ldots, x_n)}$$

$$\lambda + \delta = (\lambda_1 + n - 1, \lambda_2 + n - 2, \ldots, \lambda_n + 0)$$

E.g. : partition $\lambda = (k-2,0) \Rightarrow \lambda + \delta = (k-1,0)$

$$s_{(k-2,0)}(x_1,x_2) = \frac{\begin{vmatrix} x_1^{k-1} & 1 \\ x_2^{k-1} & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = x_1^{k-2} + x_1^{k-3}x_2 + x_1^{k-4}x_2^2 + \cdots + x_1 x_2^{k-3} + x_2^{k-2} = h_{k-2}(x_1,x_2)$$

Other specific case :

$$s_{(k-p,0^{p-1})}(x_1,\cdots,x_p) = \sum_{k_1=1}^{N} \cdots \sum_{k_p=1}^{N} \delta_{k,k_1+\cdots+k_p}\, x_1^{k_1-1} \cdots x_p^{k_p-1} = h_{k-p}(x_1,\cdots,x_p)$$

Correlation function & Schur polynomials

$$\overline{n_1 \cdots n_p} = \frac{(\pm 1)^p}{Z_N} \sum_{\substack{k_1,\cdots,k_p=1 \\ k_1+\cdots+k_p \leqslant N}}^{N} (\pm x_1)^{k_1} \cdots (\pm x_p)^{k_p}\, Z_{N-k_1-\cdots-k_p}$$

$$\overline{n_1 \cdots n_p} = \frac{x_1 \cdots x_p}{Z_N} \sum_{k=p}^{N} (\pm 1)^{p+k} \overbrace{Z_{N-k}}^{=h_{N-k} \text{ or } e_{N-k}} s_{\varpi_k}(x_1,\cdots,x_p)$$

where $\varpi_k = (k-p, 0^{p-1})$

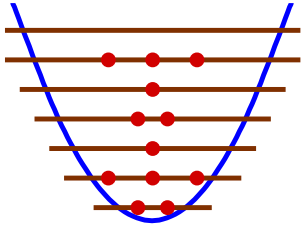
Plan

1. Introduction and motivations

2. Identical particles & a glimpse on the theory of symmetric fcts
   - Background : Canonical & Grand canonical ensembles
   - Occupation numbers
   - Symmetric functions

3. Application : N Bosons in a 1D harmonic well

One body spectrum : $\varepsilon_n = n\omega\,, \qquad n \in \mathbb{N}$



Partition function for N bosons :

$$Z_N = \prod_{n=1}^{N} (1 - e^{-n\beta\omega})^{-1} \quad \leftrightarrow \quad \begin{cases} N \text{ independent bosonic modes with} \\ \text{frequencies } \Omega_n = n\omega\,, \quad n = 1, \cdots, N \end{cases}$$

cf. problem 11.1 p. 254

"Continuous" spectrum $\Rightarrow$ $\beta F_N = -\ln Z_N \simeq \dfrac{\mathrm{Li}_2(e^{-N\beta\omega}) - \mathrm{Li}_2(1)}{\beta\omega}$

+ classical ($N\beta\omega \ll 1$) : $\beta F_N \simeq N\left[\ln(N\beta\omega) - 1\right]$ (Maxwell-Boltzmann)

Mean occupation numbers

$$\overline{n_k} = \sum_{m=1}^{N} e^{-m\beta\varepsilon_k}\, e^{-\beta(F_{N-m}-F_N)}$$

quantum $\downarrow$ regime $\omega \ll T \ll N\omega$

$$\overline{n_k} \simeq N \int_0^1 \mathrm{d}y\, \exp\left\{ -Ny\,\beta\varepsilon_k + \underbrace{\frac{\mathrm{Li}_2(e^{-N\beta\omega}) - \mathrm{Li}_2(e^{-N\beta\omega(1-y)})}{\beta\omega}}_{\simeq \frac{e^{-N\beta\omega} - e^{-N\beta\omega(1-y)}}{\beta\omega}} \right\}$$

Two parameters

$$\begin{cases} \beta\omega \ll 1 \\ z \overset{\text{def}}{=} \frac{1}{\beta\omega}\, e^{-N\beta\omega} = \overbrace{\frac{T}{\omega}}^{\gg 1}\, N^{-T_{\mathrm{B}}/T} \end{cases}$$

Simple analytic form

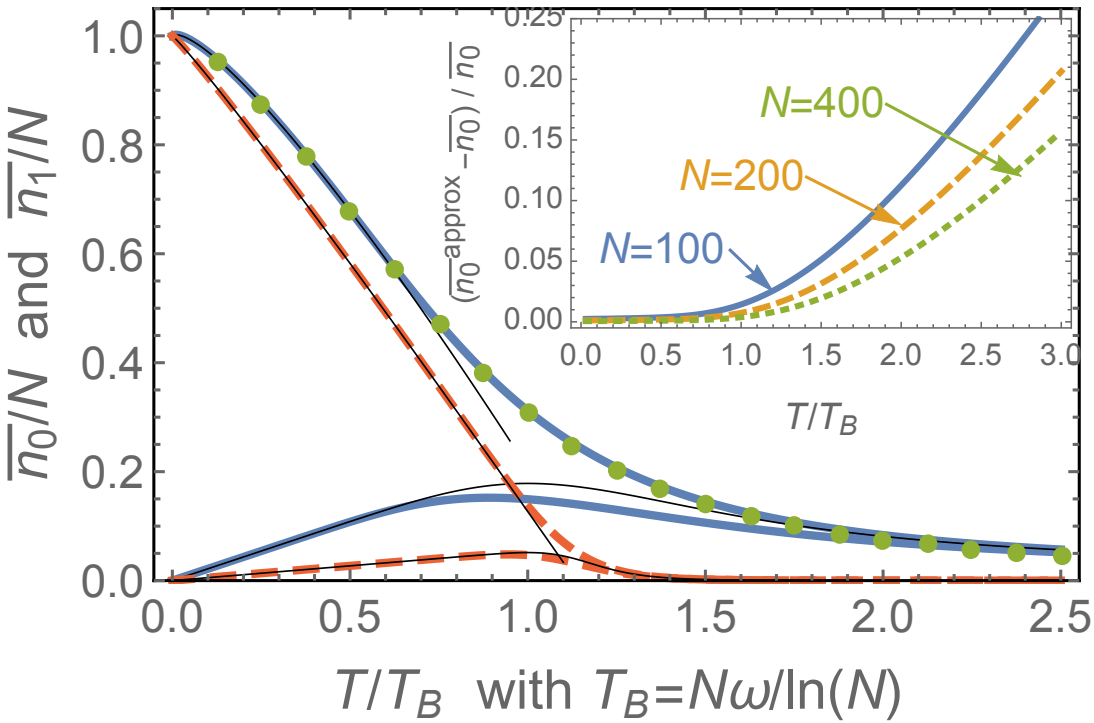
$$\boxed{\overline{n_k} \simeq \frac{1}{\beta\omega}\, z^k e^z\, \Gamma(-k, z)}$$

where $\Gamma(x,z) = \int_x^\infty \mathrm{d}t\, t^{z-1}\, e^{-t}$

Regime $\omega \ll T \ll N\omega$:

$$\frac{\overline{n_0}}{N} \simeq 1 - \frac{T \ln(e^\gamma T/\omega)}{N\omega}$$

$$\overline{n_k}^{(\text{interpolation})} \approx \frac{T}{\varepsilon_k + T\, e^{-N\omega/T}} \quad \Rightarrow \quad \overline{n_k}\big|_{\max} \simeq \frac{N/k}{\ln(N/k)}$$



$N = 10^2$ & $N = 10^8$

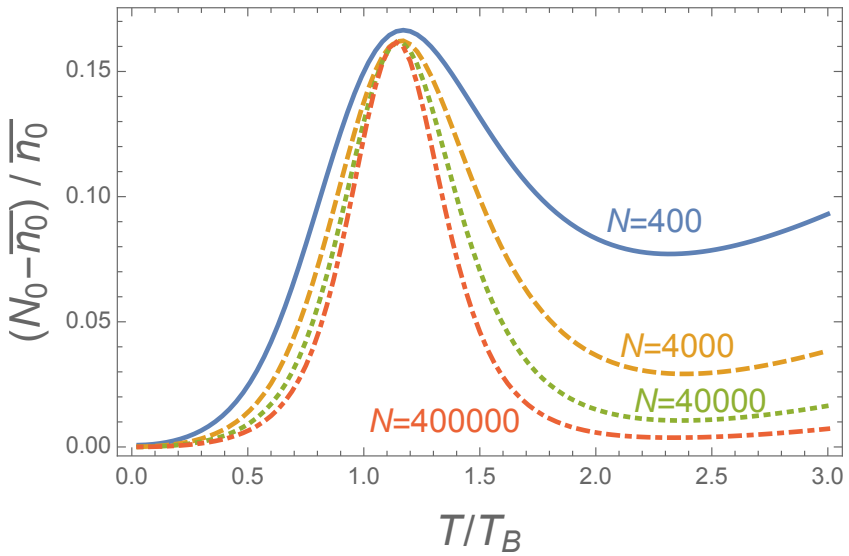
Comparison with previous results : condensate fraction

"Grand canonical approach" :

$$N - \frac{T}{\omega}\,\ln(T/\omega) = N_0 - \frac{T}{\omega}\,\psi\left(1 + \frac{T}{N_0\omega}\right)$$

Ketterle & van Druten, PRA **54** (1996)
Petrov, Gangardt & Shlyapnikov, J. Phys. IV France **116** (2004)

compare $\overline{n_0}$ (canonical) and N_0 ("grand canonical approach") :



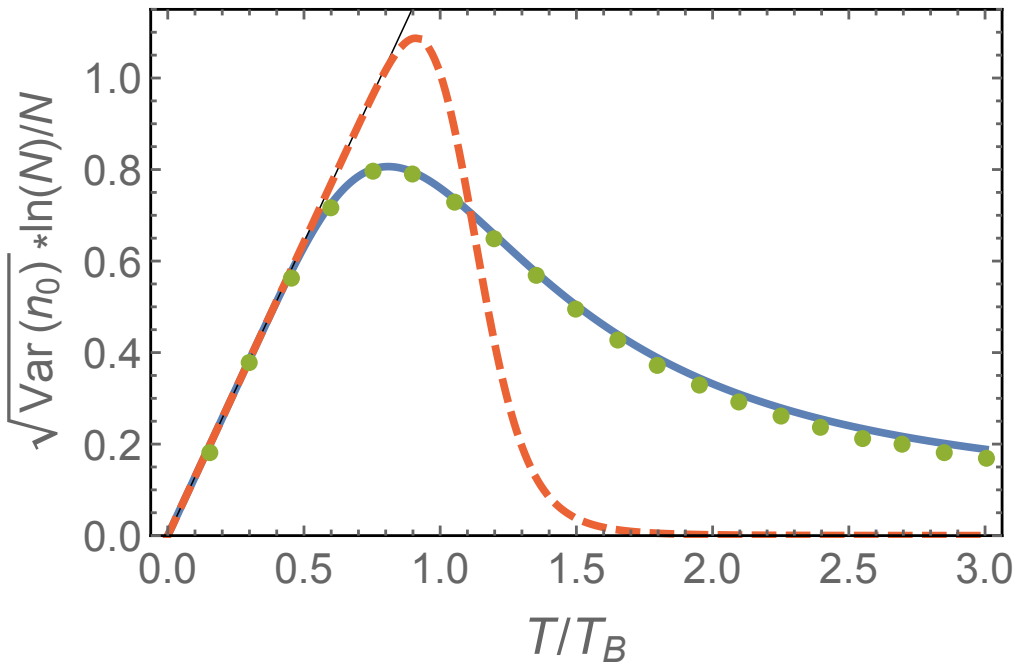
Fluctuations : $\mathrm{Var}(n_k)$ for $\omega \ll T \ll N\omega$

- $T \ll T_{\mathrm{B}}$:

$$\mathrm{Var}(n_0) \simeq \frac{1}{6}\left(\frac{\pi T}{\omega}\right)^2 \qquad \text{(Wilkens \& Weiss, 1997)}$$

$$\mathrm{Var}(n_k) \simeq \left(\overline{n_k}\right)^2 \simeq \left(\frac{T}{\varepsilon_k}\right)^2 = \left(\frac{T}{k\omega}\right)^2$$

- $T \sim T_{\mathrm{B}}$: $\mathrm{Var}(n_0)\big|_{\max} \simeq \dfrac{\pi^2 N^2}{6\ln^2 N}\left[1 - \dfrac{(\ln\ln N)^2}{\ln N} + \mathcal{O}\left(\dfrac{\ln\ln N}{\ln N}\right)\right]$



$N = 10^2$ & $N = 10^8$

Higher moments & distribution for $\omega \ll T \ll N\omega$

$$\overline{n_k^r} = \sum_{m=1}^{N} a_k(r)\, e^{-m\beta\varepsilon_k}\, e^{-\beta(F_{N-m}-F_N)} \qquad \text{with } a_k(r) = k^r - (k-1)^r$$

$$\downarrow \omega \ll T \ll N\omega$$

$$\overline{n_k^r} \simeq N^{r+1}\int_0^1 \mathrm{d}y \left[y^r - \left(y - \frac{1}{N}\right)^r\right] \exp\left[-Ny\,\beta\varepsilon_k + \frac{\mathrm{Li}_2(e^{-N\beta\omega}) - \mathrm{Li}_2(e^{-N\beta\omega(1-y)})}{\beta\omega}\right]$$

Characteristic function

$$z = \frac{e^{-N\beta\omega}}{\beta\omega}$$

$$\overline{e^{\alpha\,\beta\omega\,n_k}} \simeq 1 + \frac{1 - e^{-\alpha\,\beta\omega}}{\beta\omega}\, z^{k-\alpha} \int_z^{1/(\beta\omega)} \mathrm{d}u\, u^{\alpha-k-1} \exp\left\{\frac{\mathrm{Li}_2(\beta\omega z) - \mathrm{Li}_2(\beta\omega u)}{\beta\omega}\right\}$$

$$\begin{cases} 1/(\beta\omega) \to \infty \\ \mathrm{Li}_2(x) \simeq x \end{cases}$$

$$\boxed{\overline{e^{\alpha\,\beta\omega\,n_k}} \simeq 1 + \alpha\, e^z\, z^{k-\alpha}\, \Gamma(\alpha - k, z) \equiv G_{k,z}(\alpha)}$$

Moments of n_k

$$G_{k,z}(\alpha) = 1 + \alpha\, e^z\, z^{k-\alpha}\, \Gamma(\alpha - k, z) = \sum_{r=0}^{\infty} \frac{\mu_r(z,k)}{r!}\alpha^r \qquad \text{with } z = \frac{e^{-N\beta\omega}}{\beta\omega}$$

$$\overline{n_k^r} \simeq \frac{\mu_r(z,k)}{(\beta\omega)^r}$$

We get

$$\mu_1(z,k) = z^k e^z\, \Gamma(-k,z)$$

$$\mu_r(z,k) = r!\, e^z\, G^{r+1,0}_{r,r+1}\left(z \,\middle|\, \begin{matrix} 1+k,\ldots,1+k \\ 0,k,\ldots,k \end{matrix} \right)$$

Distribution of $\nu \simeq \beta\omega\, n_k$

$$Q_{k,z}(\nu) = \theta_{\mathrm{H}}(\nu)\, e^z\, (k + z e^\nu)\, \exp\{-k\nu - z e^\nu\}$$

Comparison with the grand-canonical result :

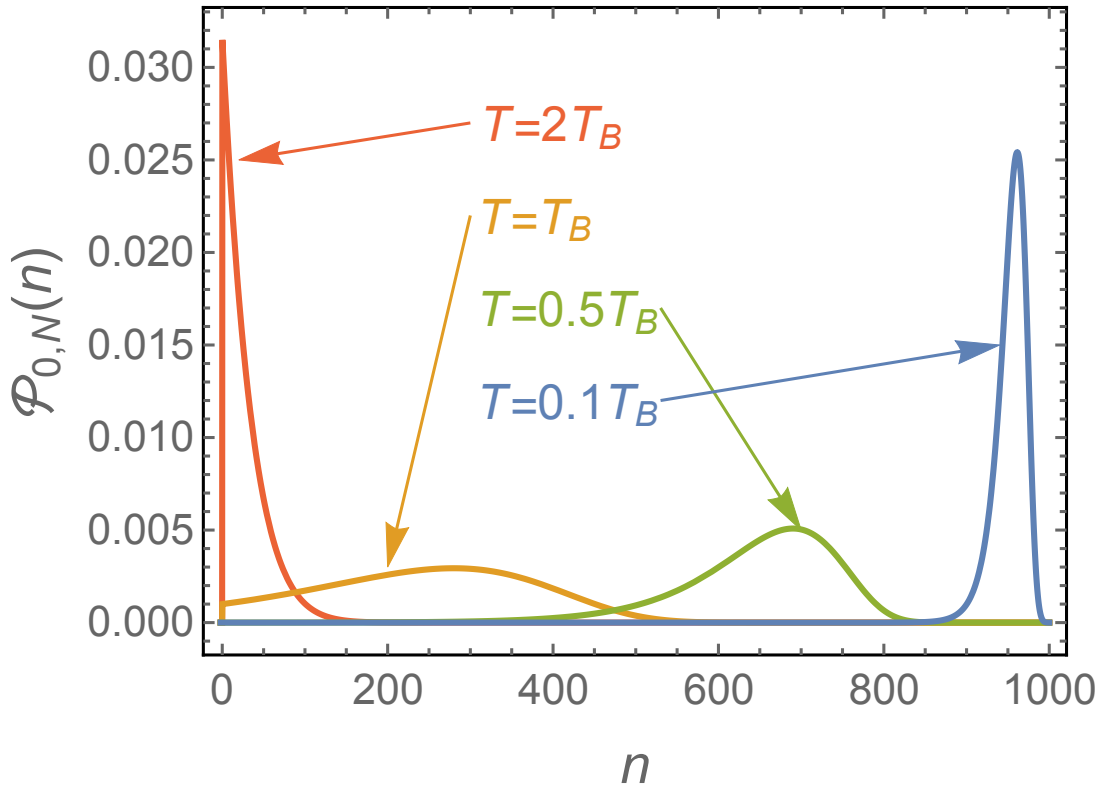
$$\mathscr{P}^{\mathrm{g}}_k(n) \propto \varphi^n e^{-n\beta\varepsilon_k} \qquad \overset{\varphi^{\mathrm{c}}=1-e^{-N\beta\omega}}{\longrightarrow} \qquad Q^{\mathrm{g}}_k(\nu) \propto e^{-(k+z)\,\nu}$$

Distribution of $\theta = \nu + \ln z \simeq \beta\omega\, n_0 + \ln z$

$$\boxed{F_z(\theta) \equiv Q_{0,z}(\nu) = \theta_{\rm H}(\theta - \ln z)\, \exp\left\{z + \theta - e^{\theta}\right\}}$$

Truncated Gumbel law

$$\mathscr{P}_{0,N}(n) \simeq \frac{\omega}{T} F_z\left(\frac{\omega}{T}\left(n - N + \frac{T}{\omega}\ln(T/\omega)\right)\right), \qquad z = \frac{T}{\omega} N^{-T_{\rm B}/T}$$



Thank you.

O. Giraud, A. Grabsch & C. Texier,
Correlations of occupation numbers in the canonical ensemble and application to BEC in a 1D harmonic trap,
Phys. Rev. A **97**, 053615 (2018).

Revenons à Gilles...

Cargèse, 2008

Weak localization in multiterminal networks of diffusive wires

Christophe Texier[1,2] and Gilles Montambaux[2]

[1] *Laboratoire de Physique Théorique et Modèles Statistiques. Université Paris-Sud, Bât. 100, F-91405 Orsay Cedex, France.*
[2] *Laboratoire de Physique des Solides. Université Paris-Sud, Bât. 510, F-91405 Orsay Cedex, France.*

(Dated: February 20, 2004)

We study the quantum transport through networks of diffusive wires connected to reservoirs in the Landauer-Büttiker formalism. The elements of the conductance matrix are computed by the diagrammatic method. We recover the combination of classical resistances and obtain the weak localization corrections. For arbitrary networks, we show how the cooperon must be properly weighted over the different wires. Its nonlocality is clearly analyzed. We predict a new geometrical effect that may change the sign of the weak localization correction in multiterminal geometries.

PACS : 73.23.-b ; 73.20.Fz ; 72.15.Rn

How to increase transmission with weak localization ? This is one the question addressed in this letter devoted to a general description of transport in networks of quasi-one-dimensional weakly disordered wires. At the classical level, the network is equivalent to a network of classical resistances, which gives the dominant contribution to the conductances. Additionally, there exists a small correction, the "weak localization" (WL) correction, originating from quantum interferences. For the conductivity, it reads [1, 2] : $\Delta\sigma(\vec{r}) = -\frac{e^2}{\pi} P_c(\vec{r},\vec{r})$ (with $\hbar = 1$, without spin factor), where the cooperon P_c is the contribution from pairs of time reversed trajectories to the return probability. Networks are particularly well suited to study interference effects and the first experimental studies of WL in such systems, performed on honeycomb lattices [3], showed the oscillations predicted by Altshuler-Aronov-Spivak (AAS) [4]. These experimental results were well fitted by the theory of Douçot & Rammal (DR) [5] whose starting point is a uniform integration of $\Delta\sigma(\vec{r})$ over the wires of the network [6]. However, although this procedure is meaningful for a translation invariant system, it is not valid in general for networks. In this letter we demonstrate that the correct expression of the WL correction to the resistance $\mathcal{R}$ of a network of wires $(\mu\nu)$ of lengths $l_{\mu\nu}$ and section s is instead :

$$\Delta\mathcal{R} = \sum_{(\mu\nu)} \frac{\partial\mathcal{R}^{\rm cl}}{\partial R^{\rm cl}_{\mu\nu}} \Delta R_{\mu\nu} \text{ with } \frac{\Delta R_{\mu\nu}}{R^{\rm cl}_{\mu\nu}} = \int_{(\mu\nu)} \mathrm{d}\vec{r}\, \frac{-\Delta\sigma(\vec{r})}{s\, l_{\mu\nu}\, \sigma_0} \tag{1}$$

where $\mathcal{R}^{\rm cl}$ is the classical resistance obtained from the classical laws of combination of resistances $R^{\rm cl}_{\mu\nu}$. The sum runs over all wires $(\mu\nu)$. σ_0 is the Drude conductivity. This result has a simple structure : it could be obtained from small variations $\Delta R_{\mu\nu}$ in the classical expression $\mathcal{R}^{\rm cl}$ of the resistance. However it is highly non trivial since, due to nonlocality, it is not possible to get a quantum formula for the resistance of the network as a function of quantum resistances of the wires. It is in fact not even possible to define a quantum resistance for a wire, independently of the whole network. Eq. (1) shows that a uniform integration of the cooperon is applicable only to regular networks in which the weights of the wires $\partial\mathcal{R}^{\rm cl}/\partial R^{\rm cl}_{\mu\nu}$ are all equal, like in the experiments of Ref. [3]. We show that in a multiterminal geometry, some of these weights can change in sign. This can lead to a *change in sign of the weak localization correction.*

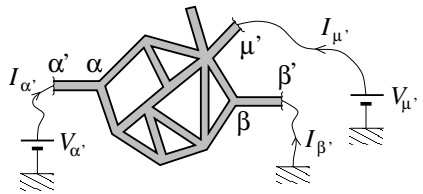


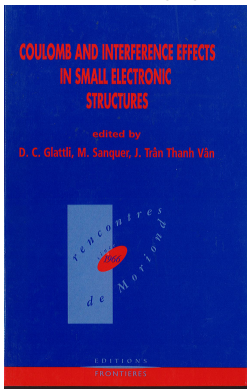
Figure 1: A network of diffusive wires. The network is connected at the vertices α', β' and μ' to external reservoirs through which some current is injected in the network.

As on figure 1, we consider networks that can be connected to external contacts (here the contacts α', β' and μ'). The transmission probability between two contacts (the conductance matrix element, up to a factor e^2/h), when averaged over the disorder, can be written as $\langle T_{\alpha'\beta'}\rangle = T^{\rm cl}_{\alpha'\beta'} + \Delta T_{\alpha'\beta'} + \cdots$, where the first term is the classical result (Drude conductance) and the second is the WL correction. Our aim is to give a systematic way to compute the WL contribution in terms of matrices encoding the information on the network (topology, lengths of the wires, magnetic fluxes). The most natural approach to describe transport in networks is the Landauer-Büttiker approach. However some difficulties related to the question of current conservation are more conveniently overcome in the Kubo formulation. We first recall some features of the transport theory of weakly disordered metals in the Kubo approach and eventually

Conférences...

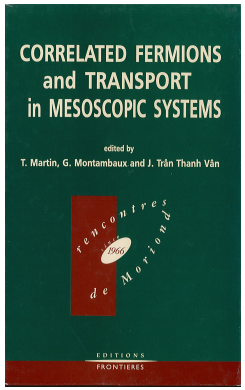
Moriond - / -

1994



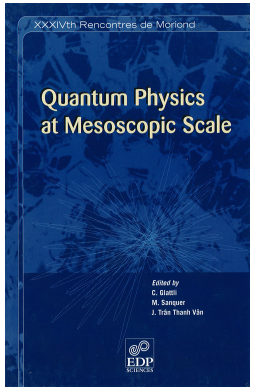
Glattli-Sanquer

1996



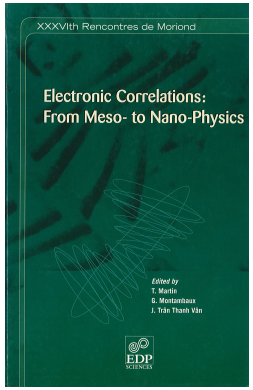
Martin-Montambaux

1999



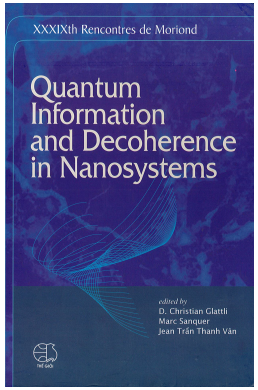
Glattli-Sanquer

2001



Martin-Montambaux

2004

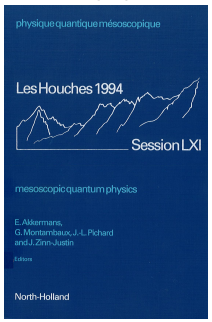


Glattli-Sanquer

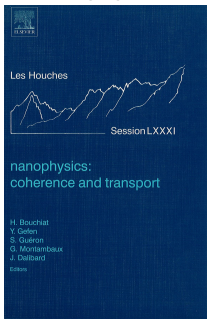
...

Les Houches

1994



2004



- Ecole de physique méso

  gestation : 20 mois

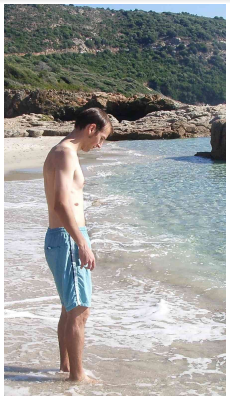
  date de naissance : 6-18 oct 2008

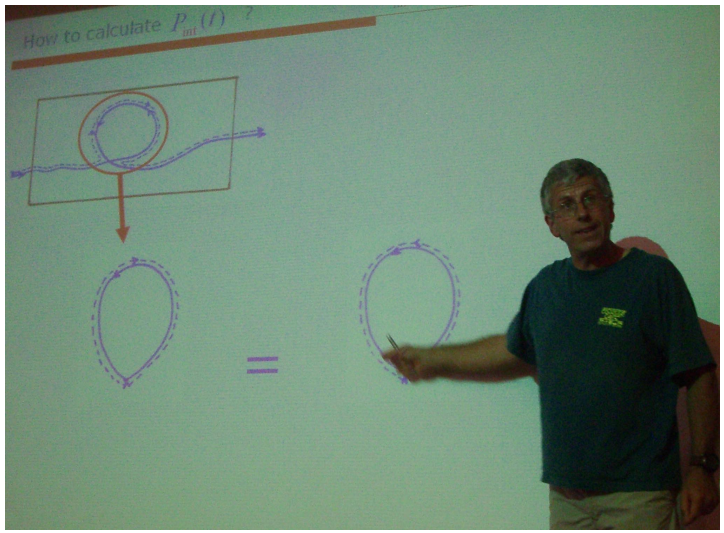
- Andrea Texier

  gestation : 9 mois

  date de naissance : 18 oct 2008 !

Cargèse (2008)





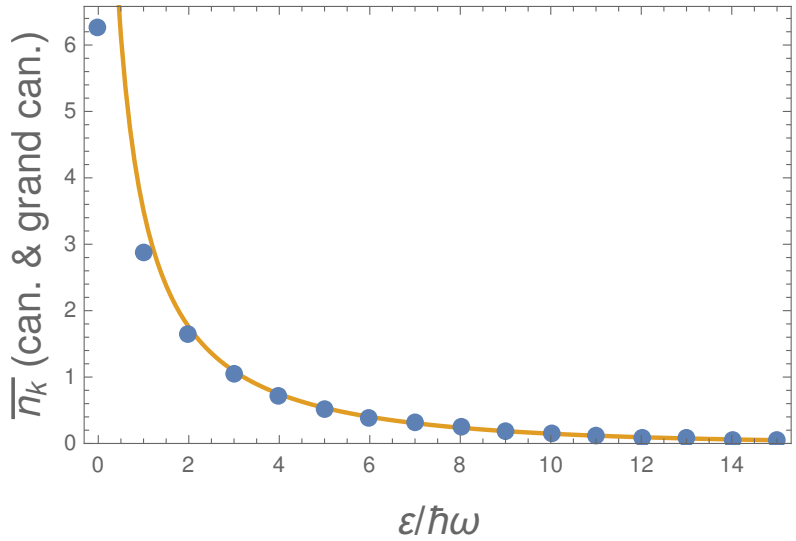
Les Houches, 2004

Merci Gilles !

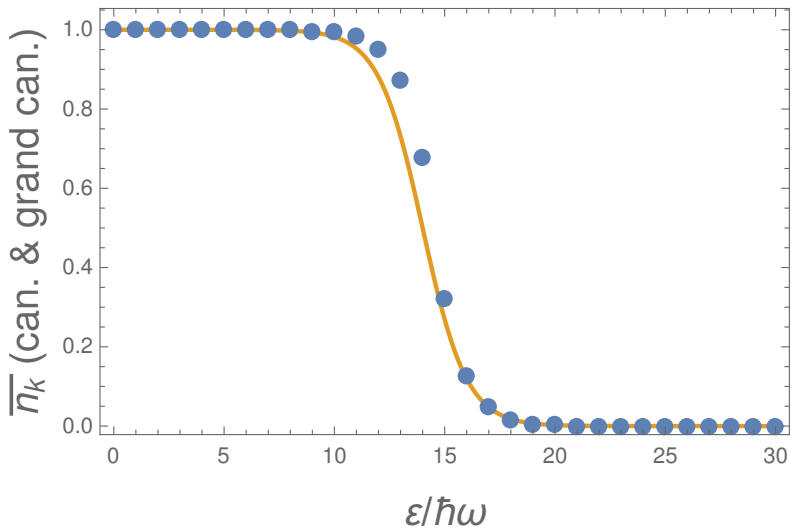
E.g. : 1D harmonic oscillator with $N = 15$ particles

- BE & FD distributions ($\overline{n_\lambda}^{\mathrm{g}}$ with $\varphi \to \varphi^{\mathrm{c}} = Z_{N-1}/Z_N$)

BE for $T = 5\omega$



FD for $T = \omega$



- Correlations (bosons)

$$\frac{\overline{n_k n_0} - \overline{n_k} \times \overline{n_0}}{\sqrt{\mathrm{Var}(n_k)\mathrm{Var}(n_0)}} \simeq -\sqrt{\frac{\mathrm{Var}(n_k)}{\mathrm{Var}(n_0)}} \simeq -\frac{\sqrt{6}}{\pi\, k} \quad \text{for } \omega \ll T \ll T_{\mathrm{B}}$$